ELSEVIER Journal of Geometry and Physics 29 (1999) 260–282

JOURNAL OF GEOMETRY AND PHYSICS

On the configuration manifold of a liquid bridge

Hans-Peter Kruse

Zentrum Mathematik, Technische Universität München, Arcisstrasse 21, D-80290 München, Germany

Received 30 March 1998; received in revised form 7 July 1998

Abstract

We show that the configuration space of an inviscid incompressible liquid bridge connecting two parallel plates has the structure of a Hilbert manifold. To construct this manifold structure we follow the general strategy of Ebin and Marsden [Groups of diffeomorphisms and the motion of an incompressible fluid, Ann. Math. 92 (1970) 102–163], where a manifold structure has been introduced for an inviscid incompressible fluid which completely fills a domain with smooth boundary.

The fact that the liquid bridge has a non-smooth boundary requires extra considerations. In particular, we show how the use of Hodge theory as in the above mentioned reference can be avoided in the case of liquid bridges. © 1999 Elsevier Science B.V. All rights reserved.

Subj. Class.: Classical field theory
1991 MSC: 58D; 58F; 76A; 76C
Keywords: Manifolds of mappings; Infinite-dimensional Hamiltonian systems; Dynamics of ideal fluids

1. Introduction

In this paper we continue the study of ideal liquid drops trapped between parallel plates begun in [6,8,9]. As in our previous work we stress the role the Hamiltonian structure of the drop equations plays in the analysis of its motion. This structure has been used in the papers cited above to derive stability results for rigidly rotating fluid drops by applying a variant of the *energy–momentum method*, which was introduced by Lewis, Marsden, Simo and their collaborateurs to analyse the stability of relative equilibria in Hamiltonian systems with symmetry (compare the articles [11,13] and the book by Marsden [12].) The instantaneous position of the liquid drop is described by elements of the configuration space of the system. These are maps of a fixed reference configuration into $\mathbb{R}^3$ encoding the instantaneous position of each fluid element of the reference configuration.

A main subject of this paper will be to closely analyse the structure of the configuration space for the liquid drop between two plates with fixed contact lines. The importance of

0393-0440/99/$19.00
PII S0393-0440(98)00040-0

configuration spaces stems from the fact that they are usually the objects introduced first in the construction of a Hamiltonian structure for the physical system under investigation. A main objective of this paper is to show that it is possible to define a Hilbert manifold structure on the set of drop configurations. Such a manifold structure has been introduced by Ebin and Marsden [2] on the space of configurations of an ideal liquid which completely fills a vessel with smooth boundary. In that case the fluid has no free boundary and the configuration space has the structure of an infinite-dimensional group which carries a Riemannian metric. Ebin and Marsden use the differentiable structure to prove a short time existence result for three-dimensional ideal fluid flow without free boundary. This flow is along geodesics of the Riemannian structure on the configuration manifold.

Configuration manifolds for ideal fluids with free boundaries have been introduced at a formal level in [10] for the case of a free liquid drop and in [6] for liquid drops trapped between plates. The question if it is possible to introduce a Hilbert manifold structure as in the case of a fluid without any free boundary has not been pursued in these papers. We will show that it can be answered in the positive for the drop model with fixed contact lines by using the techniques introduced by Ebin and Marsden. However, some non-trivial modifications have to be made: In particular, Ebin and Marsden make use of the Hodge decomposition theorem for forms on manifolds with smooth boundaries. In our case the drop boundary is not smooth, because it has corners at the intersection points of the free surface with the two plates. As it turns out in our case the application of the Hodge decomposition theorem can be circumvented by making use of the fact that parts of the drop boundary are free. An interesting problem that still seems to be open and which already has been alluded to in the original work of Ebin and Marsden is to generalize their results to fluids without free boundary which completely fill a vessel with corners.

Different from the model considered in [6] we assume the contact lines in which the free surface of the drop meets the plates to be fixed throughout the motion. By this we mean that the curves are mapped onto themselves under the drop motion. We do not ask them to stay pointwise fixed. Furthermore, we assume that the plates are completely wetted by the drop, i.e. the contact lines are along the rims of the plates. This model has been proposed in the case of axisymmetric potential flow of an ideal fluid by Eidel [3]. We present some more details of Eidel's model below. Note that we assume in particular that the drop never loses contact with the two plates. A drop trapped between two plates is also called a *liquid bridge*.

Whereas the contact lines are assumed to be fixed throughout the motion, the angles in which the free surface meets the plates are allowed to vary. In contrast to this we assumed in our previous work that the contact lines are free to move along infinitely extended plates and that the contact angles stay put throughout the motion and equal to the angles in which the free surface meets the plates in a static equilibrium configuration of the drop (see [4] for the mathematical theory of liquid masses with free boundary at rest).

The methods of this paper can be modified to define a manifold structure on the configuration space of a liquid bridge with moving contact lines in case the contact angles at the two plates are different from zero. This is explained in more detail at the end of Section 3 after we have introduced the manifold structure on the configuration space of a drop with fixed contact lines. There is an important difference between these two liquid bridge models: As explained above, the drop with moving contact lines is assumed to be trapped between two infinitely extended plates. Therefore the possibility arises that the free surface of the drop might hit one of the plates when the drop is moving. One has to exclude these configurations if one wants to have the structure of a smooth manifold *without boundary* on the configuration space of a drop with moving contact lines. Note that this situation can not arise in the model of the drop with fixed contact lines between plates considered in this paper because the contact lines are assumed to be along the rims of the finitely extended plates.

The paper is organized as follows. In Section 2 we describe the geometry of our drop model and its dynamics under the influence of surface tension forces. As mentioned above, the equations of motion for the drop are a generalization of those given by Eidel [3] for axisymmetric irrotational motion of an inviscid drop between two plates. We show that the equations of motion can be written in Hamiltonian form by introducing a configuration manifold, a phase space and a Poisson bracket, which is defined for a certain class of admissible real-valued functions on phase space. In particular the total energy of our system which is the sum of the kinetic energy and a surface energy, is an admissible function. This function is the Hamiltonian for the drop motion. The discussion in Section 2 in on a formal level, i.e. the differentiability properties of the objects introduced are not specified.

The differentiable structure of the Lagrangian configuration space is the topic of Section 3, which is the main section of this paper. To construct the manifold structure we make use of the strategy in [2]. As mentioned above, special care has to be taken of the intersection points of the free drop surface with the two plates. For transparency we treat the two-dimensional case only and briefly point out which modifications have to be made to treat the three-dimensional case when this is not entirely obvious. At the end of this section we sketch how a manifold structure can be defined on the configuration space of a drop with moving contact lines. In Section 4 we summarize results of the companion paper [7]. There, a variant of the energy–momentum method designed to analyse the stability of relative equilibrium solutions is used to study stability and bifurcation behaviour of rigidly rotating cylindrical drops. These are solutions to the equations of motion for arbitrary values of the angular velocity. The analysis is very much in the spirit of the one given in [9], where trapped ideal fluid drops with *movable* contact lines have been studied.

2. The equations of motion and their Hamiltonian structure

Let x, y and z denote cartesian coordinates in $\mathbb{R}^3$ and let $d, h \in \mathbb{R}^+$. We assume that the two plates bounding the liquid drop are given by

$$P_0 = \{(x, y, z)|x^2 + y^2 \leq d, z = h\},$$

and

$$P_1 = \{(x, y, z)|x^2 + y^2 \leq d, z = 0\}.$$

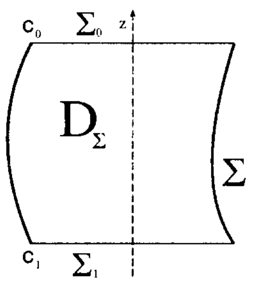

Fig. 1. The drop profile.

Let c_i denote the boundary curve of the disk P_i, $i = 0, 1$. As a reference configuration for the drop we choose the closure $\bar{E}$ of the cylinder

$$E = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 < d^2, 0 < z < h\}.$$

Let

$$F_E = \{(x, y, z) \in \mathbb{R}^3, x^2 + x^2 = d^2, 0 \leq z \leq h\}.$$

denote the lateral boundary of $\bar{E}$. Let V denote the vector space of differentiable real-valued functions on F_E and define $U \subseteq V$ by

$$U = \{g \in V | d + g(p) > 0 \text{ for } p \in F_E \text{ and } g(p) = 0 \text{ for } p \in c_0 \cup c_1\}.$$

Let

$$r(x, y, z) = \frac{1}{\sqrt{x^2 + y^2}}(x, y, 0), \tag{1}$$

for $(x, y, z) \in \mathbb{R}^3$, $x^2 + y^2 \neq 0$. For $g \in U$ and $p \in F_E$ let

$$\bar{g}(p) = p + g(p)r(p).$$

A drop configuration is specified by the position of the free boundary Σ of the drop (see Fig. 1). Σ is a two-dimensional manifold which connects the two plates, Let Σ_i denote the contact-surface of the drop and plate P_i, $i = 0, 1$. We assume that

$$\Sigma_0 = \{(x, y, z) | x^2 + y^2 \leq d, z = h\} = P_0,$$

and

$$\Sigma_1 = \{(x, y, z) | x^2 + y^2 \leq d, z = 0\} = P_1.$$

(The plates are completely covered by the drop.). Let D_Σ denote the region enclosed by Σ, Σ_1 and Σ_2, i.e. the region occupied by the drop. In this section we assume that the free drop-boundary Σ is the graph of a real-valued function on the free boundary F_E of the reference configuration $\bar{E}$. More precisely, the *Eulerian configuration space* for the drop motion is

$$\mathcal{M} = \{\Sigma \mid \Sigma = \operatorname{Im} \bar{g} \text{ for some } g \in U \text{ and } D_\Sigma = \eta(\bar{E}) \text{ for some } \eta \in \mathrm{Emb}^*_{\mathrm{vol}}(\bar{E}, \mathbb{R}^3)\},$$

where $\mathrm{Emb}^*_{\mathrm{vol}}(\bar{E}, \mathbb{R}^3)$ is the set of volume-preserving embeddings of the reference cylinder $\bar{E}$, which fix the top and the base of $\bar{E}$. In particular, the contact lines c_0 and c_1 stay fixed setwise. As mentioned in Section 1, the discussion in the section is on a formal level. Differentiability properties of elements of $\mathrm{Emb}^*_{\mathrm{vol}}(\bar{E}, \mathbb{R}^3)$ will be discussed in the next section.

$\mathrm{Emb}^*_{\mathrm{vol}}(\bar{E}, \mathbb{R}^3)$ is the *Lagrangian configuration space* $\mathcal{C}$ for the drop with fixed contact lines. Let $\mathcal{G}$ denote the group of volume-preserving diffeomorphims of the reference configuration $\bar{E}$ which keep c_0 and c_1 fixed. Let $\mathcal{C}'$ denote the set of those $\eta \in \mathcal{C}$ having the property that the free boundary of $\eta(\bar{E})$ can be written as $\bar{g}(F_E)$ for some $g \in U$. Then it is easy to check that

$$\mathcal{M} \cong \mathcal{C}'/\mathcal{G},$$

in the sense that there exits a natural bijective mapping between $\mathcal{M}$ and $\mathcal{C}'/\mathcal{G}$. Elements of the tangent bundle $T\mathcal{C}$ are pairs (η, μ), where $\eta \in \mathcal{C}$ and $\mu(p) = (\mathrm{d}/\mathrm{d}\lambda)|_{\lambda=0} c_\lambda(p)$, $c_\lambda \in \mathcal{C}$, $c_0 = \eta$. Then

$$\begin{aligned} \operatorname{div}(\mu \circ \eta^{-1}) &= 0 \\ \langle \mu \circ \eta^{-1}, (0, 0, 1)^{\mathrm{T}} \rangle &= 0 \quad \text{on } \Sigma_i, i = 0, 1, \\ \langle \mu \circ \eta^{-1}, n_i \rangle &= 0 \quad \text{on } c_i, i = 0, 1. \end{aligned}$$

Here $n_i : c_i \to \mathbb{R}^3$ denotes the vector field of outer unit normal vectors to the curve c_i in plate P_i, $i = 0, 1$. Let $n : \Sigma \to= \mathbb{R}^3$ denote the vector field of outer unit normal vectors to the free boundary Σ of the drop. The tangent space $T_\Sigma \mathcal{M}$ to an element Σ of the Eulerian configuration space $\mathcal{M}$ can be identified with the set

$$V_\Sigma = \left\{ f : \Sigma \to \mathbb{R} \middle| \int_\Sigma f \langle n, r \rangle \, \mathrm{d}A = 0, \; f = 0 \text{ on } c_0 \cup c_1 \right\}.$$

The phase space in the Lagrangian description of our problem is the tangent bundle $T\mathcal{C}$ of the Lagrangian configuration space $\mathcal{C}$. Phase space in the Eulerian description is the space

$$\mathcal{N} = \{(\Sigma, v) \mid \Sigma \in \mathcal{M}, v : D_\Sigma \to \mathbb{R}^3, \operatorname{div} v = 0, \langle v, (0, 0, 1)^{\mathrm{T}} \rangle = 0 \text{ on } \Sigma_0 \cup \Sigma_1, \quad \text{and } \langle v, n_i \rangle = 0 \text{ on } c_i, i = 0, 1\}.$$

Then

$$T\mathcal{C}'/\mathcal{G} \cong \mathcal{N}.$$

Now we describe the Poisson structure in $\mathcal{N}$. As in [6], this structure can be derived by a Marsden–Weinstein reduction from a Poisson structure on the Lagrangian phase space $T\mathcal{C}$.

First we describe the class of *admissible functions* $\mathcal{D}$, i.e. the set of real-valued functions on phase space, for which the Poisson bracket will be defined.

We say a function $F : \mathcal{N} \to \mathbb{R}$ has a functional derivative with respect to Σ in $(\Sigma, v) \in \mathcal{N}$, if

$$DF(\Sigma, v)\delta\Sigma = \int_{\Sigma} \frac{\delta F}{\delta \Sigma}(\Sigma, v)\delta\Sigma \,\mathrm{d}A,$$

for some function $(\delta F/\delta\Sigma)(\Sigma, v) : \Sigma \to \mathbb{R}$. We say a function $F : \mathcal{N} \to \mathbb{R}$ has a functional derivative with respect to v in $(\Sigma, v) \in \mathcal{N}$, if

$$DF(\Sigma, v)\delta v = \int_{D_\Sigma} \left\langle \frac{\delta F}{\delta v}(\Sigma, v), \delta v \right\rangle \mathrm{d}V,$$

for a vector field $(\delta F/\delta v)(\Sigma, v) : D_\Sigma \to \mathbb{R}^3$, satisfying

$$\operatorname{div}\left(\frac{\delta F}{\delta v}(\Sigma, v)\right) = 0 \quad \text{and} \quad \left\langle \frac{\delta F}{\delta v}(\Sigma, v), (0, 0, 1)^{\mathrm{T}} \right\rangle = 0 \quad \text{on } \Sigma_0 \cup \Sigma_1.$$

Let $r : \mathbb{R}^3 \backslash \{(x, y, z) | x^2 + y^2 = 0\} \to \mathbb{R}^3$ be defined as in (1). As explained above, $n : \Sigma \to \mathbb{R}^3$ is the vector field of outer unit normal vectors to Σ. The Poisson bracket $\{F, H\} : \mathcal{N} \to \mathbb{R}$ of two functions $F, H \in \mathcal{D}$ is defined by

$$\begin{aligned} \{F, H\}(\Sigma, v) = &\int_{D_\Sigma} \left\langle \nabla \times v, \frac{\delta F}{\delta v} \times \frac{\delta H}{\delta v} \right\rangle \mathrm{d}V \\ &+ \int_{\Sigma} \left(\frac{\delta F}{\delta \Sigma} \left\langle \frac{\delta H}{\delta v}, n \right\rangle - \frac{\delta H}{\delta \Sigma} \left\langle \frac{\delta F}{\delta v}, n \right\rangle \right) \frac{1}{\langle r, n \rangle} \mathrm{d}A. \end{aligned}$$

One can check that the sum of the kinetic energy of the drop and the potential energy due to surface tension

$$H = \frac{1}{2} \int_{D_\Sigma} \|v\|^2 \,\mathrm{d}V + \tau \int_{\Sigma} \mathrm{d}A,$$

is an element of $\mathcal{D}$. Here $\tau > 0$ is the *constant of surface tension*. The density of the drop is assumed to be $\rho = 1$. One has

$$\frac{\delta H}{\delta \Sigma} = \frac{1}{2}\|v\|^2 \langle r, n \rangle + \tau\kappa \langle r, n \rangle \quad \text{and} \quad \frac{\delta H}{\delta v} = v,$$

where κ denotes the mean curvature of the surface Σ. Now we are going to derive the set of partial differential equations, which are satisfied by solution curves

$$\begin{aligned} \mathbb{R} \supseteq I &\to \mathcal{N}, \\ t &\mapsto (\Sigma_t, v_t), \end{aligned}$$

of the Hamiltonian equations

$$\dot{F} = \{F, H\} \quad \text{for all } F \in \mathcal{D}. \tag{2}$$

Eq. (2) is just a shorthand for

$$\frac{\mathrm{d}}{\mathrm{d}t}F(\Sigma_t, v_t) = \{F, H\}(\Sigma_t, v_t) \quad \text{for all } F \in \mathcal{D}.$$

By the divergence theorem one has

$$\begin{aligned}
\{F, H\} &= -\int_{D_\Sigma} \left\langle \frac{\delta F}{\delta v}, (\nabla \times v) \times v \right\rangle \mathrm{d}V + \int_\Sigma \frac{\delta F}{\delta \Sigma} \frac{\langle v, n\rangle}{\langle r, n\rangle} \mathrm{d}A \\
&\quad - \int_\Sigma \frac{1}{2}\|v\|^2 \left\langle \frac{\delta F}{\delta v}, n \right\rangle \mathrm{d}A - \tau \int_\Sigma \kappa \left\langle \frac{\delta F}{\delta v}, n \right\rangle \mathrm{d}A \\
&= \int_{D_\Sigma} \left\langle \frac{\delta F}{\delta v}, v \times (\nabla \times v) - \nabla \frac{1}{2}\|v\|^2 \right\rangle \mathrm{d}V + \int_\Sigma \frac{\delta F}{\delta \Sigma} \frac{\langle v, n\rangle}{\langle r, n\rangle} \mathrm{d}A \\
&\quad -\tau \int_\Sigma \kappa \left\langle \frac{\delta F}{\delta v}, n \right\rangle \mathrm{d}A.
\end{aligned}$$

Making use of the fact that

$$v \times (\nabla \times v) - \tfrac{1}{2}\nabla \|v\|^2 = -(v \cdot \nabla)v, \tag{3}$$

one arrives at

$$\begin{aligned}
\{F, H\} &= \int_{D_\Sigma} -\left\langle \frac{\delta F}{\delta v}, v \cdot \nabla v \right\rangle \mathrm{d}V + \int_\Sigma \frac{\delta F}{\delta \Sigma} \frac{\langle v, n\rangle}{\langle r, n\rangle} \mathrm{d}A \\
&\quad -\tau \int_\Sigma \kappa \left\langle \frac{\delta F}{\delta v}, n \right\rangle \mathrm{d}A.
\end{aligned}$$

Define

$$\begin{aligned}
p : \{(x, t) \in \mathbb{R}^4 | x \in D_{\Sigma_t}, t \in I\} &\to \mathbb{R}, \\
(x, t) &\mapsto p(x, t),
\end{aligned}$$

to be the solution, keeping t fixed, of the boundary value problem

$$\begin{aligned}
\Delta p &= -\mathrm{div}((v \cdot \nabla)v) && \text{in } D_\Sigma, \\
p &= \tau\kappa && \text{on } \Sigma, \\
\langle \nabla p, (0, 0, 1)^{\mathrm{T}} \rangle &= -\langle (v \cdot \nabla)v, (0, 0, 1)^{\mathrm{T}} \rangle && \text{on } \Sigma_0 \cup \Sigma_1.
\end{aligned} \tag{4}$$

Then,

$$-\int_\Sigma \tau\kappa \left\langle \frac{\delta F}{\delta v}, n \right\rangle \mathrm{d}A = -\int_\Sigma p \left\langle \frac{\delta F}{\delta v}, n \right\rangle \mathrm{d}A = -\int_{D_\Sigma} \left\langle \nabla p, \frac{\delta F}{\delta v} \right\rangle \mathrm{d}V,$$

and

$$\{F, H\} = \int_{D_\Sigma} \left\langle \frac{\delta F}{\delta v}, -\nabla p - (v \cdot \nabla)v \right\rangle \mathrm{d}V + \int_\Sigma \frac{\delta F}{\delta \Sigma} \frac{\langle v, n\rangle}{\langle r, n\rangle} \mathrm{d}A.$$

Thus, the equation $\dot{F} = \{F, H\}$ is equivalent to

$$\int_{D_\Sigma} \left\langle \frac{\delta F}{\delta v}, \dot{v} \right\rangle \mathrm{d}V + \int_{\Sigma} \frac{\delta F}{\delta \Sigma} \dot{\Sigma}\, \mathrm{d}A$$
$$= \int_{D_\Sigma} \left\langle \frac{\delta F}{\delta v}, -\nabla p - (v \cdot \nabla) v \right\rangle \mathrm{d}V + \int_{\Sigma} \frac{\delta F}{\delta \Sigma} \frac{\langle v, n\rangle}{\langle r, n\rangle} \mathrm{d}A. \tag{5}$$

If $F \in \mathcal{D}$ is of the form $F = \int_{D_\Sigma} f \, \mathrm{d}V$ for some function $f : \mathbb{R}^3 \to \mathbb{R}$, then

$$\frac{\delta F}{\delta \Sigma} = f \langle r, n\rangle \quad \text{and} \quad \frac{\delta F}{\delta v} = 0.$$

Because one can choose $f : \mathbb{R}^3 \to \mathbb{R}$ arbitrarily, one concludes from Eqs. (2) and (5) that

$$\dot{\Sigma} = \frac{\langle v, n\rangle}{\langle r, n\rangle}, \tag{6}$$

and that

$$\int_{D_\Sigma} \left\langle \frac{\delta F}{\delta v}, \dot{v} + (v \cdot \nabla) v + \nabla p \right\rangle \mathrm{d}V = 0 \quad \text{for all } F \in \mathcal{D}. \tag{7}$$

For fixed t, put $u = \dot{v} + v \cdot \nabla v + \nabla p$. One has

$$\operatorname{div}(u) = 0 \quad \text{and} \quad \langle u, (0, 0, 1)^{\mathrm{T}}\rangle = 0 \quad \text{on } \Sigma_0 \cup \Sigma_1,$$

using the definition of p and the fact, that v is divergence free. Define a function $F_u : \mathcal{N} \to \mathbb{R}$ by

$$F_u(\Sigma, v) = \int_{D_\Sigma} \langle u, v\rangle \, \mathrm{d}V.$$

Then $F_u \in \mathcal{D}$ and $(\delta F_u / \delta v) = u$. Using this in (7) one arrives at

$$\dot{v} + (v \cdot \nabla) v = -\nabla p. \tag{8}$$

In summary, we have shown that a curve $t \mapsto (\Sigma_t, v_t)$ in $\mathcal{N}$ satisfying (2) is a solution to the following system of partial differential equations:

$$\dot{v} + (v \cdot \nabla) v = -\nabla p, \tag{9}$$
$$\dot{\Sigma} = \frac{\langle v, n\rangle}{\langle r, n\rangle}, \tag{10}$$
$$p = \tau \kappa \quad \text{on } \Sigma, \tag{11}$$
$$\operatorname{div} v = 0, \tag{12}$$
$$\langle v, (0, 0, 1)^{\mathrm{T}}\rangle = 0 \quad \text{on } \Sigma_0 \cup \Sigma_1, \tag{13}$$
$$\langle v, n_i\rangle = 0 \quad \text{on } c_i, \quad i = 0, 1. \tag{14}$$

Here p is the solution of the boundary value problem (4). It is easy to see that the converse holds true, i.e. that a solution to (9)–(14) yields a solution to Hamilton's equation (2).

To relate our equations to those given in [3], we specialize Equations (9)–(14) to the case of potential flows with rotationally symmetric potential, i.e. we assume the velocity field v to be of the form

$$v = \nabla \Phi$$

for a real-valued function Φ which only depends on the R and Z components of cylindrical coordinates R, Φ and Z. We assume the free boundary of the drop to be rotationally symmetric and to be defined by the equation

$$R = \mathrm{d} + \zeta(Z), \tag{15}$$

i.e. the free boundary is the level set $\Psi(R, Z) = 0$ of the function $\Psi(R, Z) = R - \zeta(Z) - d$. One has

$$\nabla \psi = \frac{\partial \Psi}{\partial R} e_R + \frac{1}{R^2} \frac{\partial \Psi}{\partial \Phi} e_\Phi + \frac{\partial \Psi}{\partial Z} e_Z = e_R - \frac{\partial \zeta}{\partial Z} e_Z,$$

where e_R, e_Φ and e_Z are unit normal vectors in the R, Φ, and Z direction, respectively (compare [5]). Thus, an outer unit normal vector field to the free boundary Σ of the drop is given by

$$n = \frac{1}{\sqrt{1 + (\partial \zeta / \partial Z)^2}} \left(e_R - \frac{\partial \zeta}{\partial Z} e_Z \right).$$

Because by definition (1)$r = e_R$,

$$\langle r, n \rangle = \frac{1}{\sqrt{1 + (\partial \zeta / \partial Z)^2}},$$

and

$$\begin{aligned} \langle v, n \rangle &= \langle \nabla \Phi, n \rangle \\ &= \left\langle \frac{\partial \Phi}{\partial R} e_R + \frac{\partial \Phi}{\partial Z} e_Z, \frac{1}{\sqrt{1 + (\partial \zeta / \partial Z)^2}} \left(e_R - \frac{\partial \zeta}{\partial Z} e_Z \right) \right\rangle \\ &= \frac{1}{\sqrt{1 + (\partial \zeta / \partial Z)^2}} \left(\frac{\partial \Phi}{\partial R} - \frac{\partial \zeta}{\partial Z} \frac{\partial \Phi}{\partial Z} \right). \end{aligned}$$

Thus, Eq. (10) can be written in the form

$$\dot{\zeta} = \frac{\partial \Phi}{\partial R} - \frac{\partial \zeta}{\partial Z} \frac{\partial \Phi}{\partial Z}.$$

Using (3) in Eq. (9) yields $\nabla \dot{\Phi} + \frac{1}{2} \nabla \|\nabla \Phi\|^2 = -\nabla p$. Thus,

$$\dot{\Phi} + \tfrac{1}{2} \|\nabla \Phi\|^2 + p = c, \tag{16}$$

for some $c \in \mathbb{R}$, i.e. because of the symmetry of Φ,

$$\dot{\Phi} + \frac{1}{2} \left(\left(\frac{\partial \Phi}{\partial R} \right)^2 + \left(\frac{\partial \Phi}{\partial Z} \right)^2 \right) + p = c. \tag{17}$$

Using (11) in evaluating (17) on the free boundary Σ of the drop yields

$$\dot{\Phi} = \frac{1}{2}\left(\left(\frac{\partial \Phi}{\partial R}\right)^2 + \left(\frac{\partial \Phi}{\partial Z}\right)^2\right) + \tau\kappa = c. \tag{18}$$

Because the elements of our configuration space have fixed contact lines c_0 and c_1, one has $\zeta = 0$ for $Z = 0$ and $Z = h$, and from (13) it follows that $(\partial\Phi/\partial Z) = 0$ at $Z = 0$ and $Z = h$. Eq. (12) can be written in the form $\Delta\Phi = 0$.

In summary, we see that in case $v = \nabla\Phi$ for a rotationally symmetric potential Φ Eqs. (9)–(14) can be written in the form

$$\Delta\Phi = 0, \tag{19}$$

$$\dot{\zeta} - \frac{\partial \Phi}{\partial R} + \frac{\partial \zeta}{\partial Z}\frac{\partial \Phi}{\partial Z} = 0, \tag{20}$$

$$\dot{\Phi} + \frac{1}{2}\left(\left(\frac{\partial \Phi}{\partial R}\right)^2 + \left(\frac{\partial \Phi}{\partial Z}\right)^2\right) + \tau\kappa = c \quad \text{at } R = d + \zeta(Z), \tag{21}$$

$$\frac{\partial \Phi}{\partial Z} = 0 \quad \text{at } Z = 0, h, \tag{22}$$

$$\zeta = 0 \quad \text{at } Z = 0, h. \tag{23}$$

If $c = \tau/d$ holds, Eqs. (19)–(23) are just the equations of motion in [3] for the special case that the density of the drop is given by $\rho = 1$. Eidel asks the drop volume to be equal to that of the reference cylinder with radius d and height h. In particular, a cylinder at rest with base radius d is a solution to Eidel's equations of motion.

3. The manifold structure of the configuration space

For transparency we consider the two-dimensional case, i.e. the configuration space of a plane drop. We will point out which modifications have to be made to handle the three-dimensional case when it seems to be necessary.

To simplify notation, we assume the height of the drop to be $h = 2$ and the base radius to be $d = 1$. We choose coordinates $x, y \in \mathbb{R}^2$ such that the reference configuration (see Fig. 2) of the drop is the closure $\bar{E}$ of the set

$$E = (-1, 1) \times (-1, 1).$$

For a subset $B \neq \emptyset$ of $\mathbb{R}^2$, let $C^1(B, \mathbb{R}^2)$ denote the space of C^1-maps of $\bar{B}$ into $\mathbb{R}^2$ which can be extended to C^1-maps on a neighbourhood of $\bar{B}$.

Note, that by this definition, it makes sense to evaluate $\eta \in C^1(E, \mathbb{R}^2)$ on the boundary of E.

We will now outline our strategy to put a manifold structure on the configuration space $\mathcal{C}$ introduced in the preceding section. Let $\mathrm{Emb}^*(\bar{E})$ denote the set of embeddings of $\bar{E}$ into $\mathbb{R}^2$, which map Σ_0 and Σ_1 onto themselves and which fix the four corner points of $\bar{E}$. The degree of differentiability of the maps in $\mathrm{Emb}^*(\bar{E})$ will be specified later. We will show that

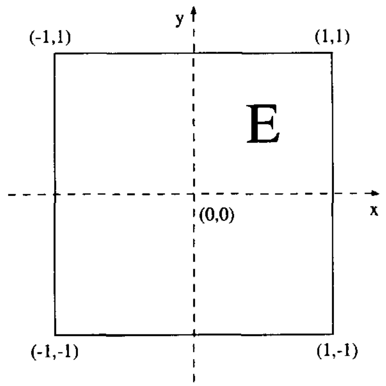

Fig. 2. The reference configuration.

$\mathrm{Emb}^*(\bar{E})$ is a submanifold of a Hilbert space of mappings. Let $\Lambda^2(\bar{E})$ denote the vector space of volume elements on $\bar{E}$. (Again, the exact differentiability properties of elements of $\Lambda^2(\bar{E})$ will be specified later). These volume elements are of the form $f\, \mathrm{d}x \wedge y$, where f is some real-valued function on $\bar{E}$. We will show that the map

$$\begin{aligned} \Psi : \mathrm{Emb}^*(\bar{E}) &\to W, \\ \eta &\mapsto \eta^* \mathrm{d}x \wedge \mathrm{d}y, \end{aligned}$$

is differentiable. Here W is a certain subspace of $\Lambda^2(\bar{E})$ to be defined later. Then we will prove that

$$\mathcal{C} = \Psi^{-1}(\mathrm{d}x \wedge \mathrm{d}y)$$

is a submanifold of $\mathrm{Emb}^*(\bar{E})$ by showing that the map $T_\eta \Psi$ is surjective at every $\eta \in \Psi^{-1}(\mathrm{d}x \wedge \mathrm{d}y)$.

Now we will fill in the details in the program outlined above. To put a manifold structure on $\mathrm{Emb}^*(\bar{E})$, we will show that $\mathrm{Emb}^*(\bar{E})$ can be considered as a submanifold of $H^s(E, \mathbb{R}^2)$ for $s > 2$. $H^s(E, \mathbb{R}^2)$ is a Hilbert space and therefore a trivial Hilbert manifold. One arrives at the same manifold structure on $H^s(E, \mathbb{R}^2)$ if one applies the well-known construction to introduce a manifold structure on a set of maps from a manifold M to a manifold N. Of course this construction is trivial in the situation at hand: For $p \in \mathbb{R}^2$ define the exponential map with respect to the standard metric on $\mathbb{R}^2$

$$\begin{aligned} \exp_p : \mathbb{R}^2 &\to \mathbb{R}^2, \\ x &\mapsto p + x. \end{aligned}$$

If one has to deal with more general Riemannian manifolds N than $\mathbb{R}^2$, the exponential map is defined only in a neighbourhood of $0 \in T_pN$. Because $\mathbb{R}^2$ is geodesically complete, the map $\exp_p$ is defined on all $T_p\mathbb{R}^2 = \mathbb{R}^2$. Ebin and Marsden [2] are dealing with general Riemannian manifolds N and to ensure that N is geodesically complete they assume that it is compact.

Local coordinates parametrizing a neighbourhood of $\eta \in H^s(E, \mathbb{R}^2)$ are given by

$$\begin{aligned} H^s(E, \mathbb{R}^2) &\to H^s(E, \mathbb{R}^2), \\ X &\mapsto (q \mapsto \eta(q) + X(q)). \end{aligned} \tag{24}$$

One has to check that the change of coordinates is well-defined and smooth. This is obvious from (24): If $\eta \in H^s(E, \mathbb{R}^2)$ can be written in two different ways $\eta = \eta_1 + X_1$ and $\eta = \eta_2 + X_2$ for $\eta_1, \eta_2, X_1, X_2 \in H^s(E, \mathbb{R}^2)$, then changing coordinates means substituting $X_2 = \eta_1 + X_1 - \eta_2$ for X_1, which is a smooth map.

As alread stated above, as a Hilbert space, $H^s(E, \mathbb{R}^2)$ is a trivial Hilbert manifold, taking the identity map $H^s(E, \mathbb{R}^2) \to H^s(E, \mathbb{R}^2)$ as a local chart. The differentiable structure one arrives at using this chart is the same one gets by using the general construction outlined above. In particular,

$$T_\eta H^s(E, \mathbb{R}^2) = H^s(E, \mathbb{R}^2).$$

Let $\mathrm{Emb}(\bar{E})$ denote the set of embeddings $\eta : \bar{E} \to \mathbb{R}^2$, where by an embedding we mean an H^s-map ζ with an inverse $\zeta^{-1} \in C^1(\zeta(\bar{E}), \mathbb{R}^2)$. As explained above, $C^1(\zeta(\bar{E}), \mathbb{R}^2)$ is the space of C^1-maps from $\zeta(\bar{E})$ to $\mathbb{R}^2$, which can be extended to C^1-maps on a neighbourhood of the closure of $\zeta(\bar{E})$. Note, that by Sobolev's embedding theorem it makes sense to evaluate elements of $H^s(E, \mathbb{R}^2)$ on $\bar{E}$. Now we will show that $\mathrm{Emb}(\bar{E})$ is a trivial submanifold of $H^s(E)$.

Lemma 1. *$Emb(\bar{E})$ is an open subset of $H^s(E)$.*

Proof. First we show that there is a neighbourhood of η in $H^s(E)$, consisting of maps ζ that map $\bar{E}$ *locally* diffeomorphic onto $\zeta(\bar{E})$.

Let V denote an open bounded neighbourhood of E in $\mathbb{R}^2$. By the theorem of Calderon–Zygmund there is a linear and continuous extension map $H^s(E) \to H^s(\mathbb{R}^2)$ (compare [14]). Composing with a restriction map yields a linear continuous extension map

$$H^s(E) \to H^s(V).$$

Since by Sobolev's embedding theorem the inclusion map $H^s(V, \mathbb{R}^2) \to C^1(V, \mathbb{R}^2)$ is continuous, we have a continuous map

$$H^s(E) \to C^1(V, \mathbb{R}^2). \tag{25}$$

Therefore, it suffices to show that for each $\eta \in \mathrm{Emb}(\bar{E})$ there is a neighbourhood $U \subseteq C^1(V, \mathbb{R}^2)$ consisting of maps that are local diffeomorphisms when restricted to $\bar{E}$. Let $\|\cdot\|$ denote the standard norm on $C^1(V, \mathbb{R}^2)$. Define

$$\begin{aligned} \sigma : V \times C^1(V, \mathbb{R}^2) &\to \mathbb{R}^2 \times C^1(V, \mathbb{R}^2), \\ (x, X)^{\mathrm{T}} &\mapsto (\eta(x) + X(x), X)^{\mathrm{T}}. \end{aligned}$$

Then

$$D\sigma(x, X) = \begin{pmatrix} D_x\eta + (D_x X) & * \\ 0 & \mathrm{Id} \end{pmatrix}$$

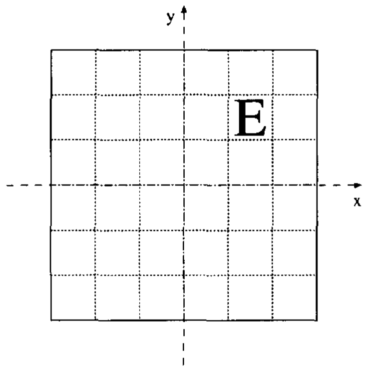

Fig. 3. Division of E in the proof that injectivity is preserved under small perturbations.

is invertible for $x \in \bar{E}$, $\|X\| < \epsilon$ and $\epsilon \in \mathbb{R}^+$ sufficiently small. The inverse $D\sigma^{-1}$ has the form

$$D\sigma^{-1} = \begin{pmatrix} (D_x\eta + (D_xX))^{-1} & -(D_x\eta + (D_xX))^{-1} \circ (*) \\ 0 & \mathrm{Id} \end{pmatrix}.$$

The inverse function theorem then guarantees the existence of a neighbourhood $B_{\delta_x}(x) \times B_{\epsilon_x}(0)$ of $(x, 0)$ in $V \times C^1(V, \mathbb{R}^2)$, such that the restriction of σ to this neighbourhood is a diffeomorphism onto its image. Since $\bar{E}$ is compact, we can cover it by a finite number of balls $B_{\delta_{x_i}}(x_i)$, $i = 1, \ldots, n$, such that

$$\begin{aligned} B_{\delta_{x_i}}(x_i) &\to \mathbb{R}^2, \\ x &\mapsto \eta(x) + X(x), \end{aligned}$$

is a local diffeomorphism for $X \in B_{\epsilon_{x_i}}(0)$. In particular,

$$\begin{aligned} &\bar{E} \to \mathbb{R}^2, \\ &x \mapsto \eta(x) + X(x), \end{aligned}$$

is a local diffeomorphism for $X \in B_{\bar{\epsilon}}(0)$, where $\bar{\epsilon} = \min_{i=1,\ldots,n} \epsilon_{x_i}$. Thus $\eta + B_{\bar{\epsilon}}(0)$ is the neighbourhood of η in $C^1(V, \mathbb{R}^2)$ we were looking for, consisting of maps ζ that map $\bar{E}$ *locally* diffeomorphic onto $\zeta(\bar{E})$.

The proof shows that we can cover E by small closed squares $S_1, \ldots, S_k$ as in Fig. 3 in such a way that $\zeta \in \eta + B_{\bar{\epsilon}}(0)$ is a diffeomorphim when restricted to any square consisting of four small squares. Define a function $M : \eta + B_{\bar{\epsilon}}(0) \to \mathbb{R}$ by

$$M(\zeta) = \min\{|\zeta(x) - \zeta(y)|, x \in S_i, y \in S_j \text{ for some } S_i, S_j \text{ with } S_i \cap S_j = \emptyset\}.$$

Because η is injective when restricted to E, one has $M(\eta) > 0$ and since the function M is continuous, one can find $\hat{\epsilon} < \bar{\epsilon}$, such that $M(\zeta) > 0$ for $\zeta \in \eta + B_{\hat{\epsilon}}(0)$. In particular, the map ζ is injective: because $M(\zeta) > 0$, the images of non-neighbouring small squares do not intersect. But images of neighbouring small squares do not intersect either, because ζ is injective on every square built from four neighbouring small squares. Therefore, $\eta + B_{\hat{\epsilon}}(0)$ consists of maps that are local diffeomorphims and injective, when restricted to E. Since the map (25) is continuous, it follows that $\mathrm{Emb}(\bar{E})$ is an open subset and therefore a trivial submanifold of $H^s(E, \mathbb{R}^2)$. Let U_η denote a neighbourhood of η in $H^s(E, \mathbb{R}^2)$ that is mapped into $\eta + B_{\hat{\epsilon}}(0)$ by the map (25). Local charts in a neighbourhood of $\eta \in \mathrm{Emb}(\bar{E})$ are given by

$$\begin{aligned} \Phi_\eta : \eta + U_\eta &\to U_\eta, \\ \zeta &\mapsto \eta - \zeta. \end{aligned} \tag{26}$$

This finishes the proof of the lemma. □

Let $\mathrm{Emb}^*(\bar{E})$ denote the set of those elements of $\mathrm{Emb}(\bar{E})$ that map Σ_0 and Σ_1 onto themselves, keeping the four corner points of $\bar{E}$ fixed.

Lemma 2. *$\mathrm{Emb}^*(\bar{E})$ is a submanifold of $\mathrm{Emb}(\bar{E})$.*

Proof. Define a subspace $X_\| \subseteq H^s(E, \mathbb{R}^2)$ by

$$\begin{aligned} X_\| = \{X \in H^s(E, \mathbb{R}^2) \mid \langle X, (0,1)^{\mathrm{T}} \rangle &= 0 \text{ on } \Sigma_0 \cup \Sigma_1, \\ &\text{and } X = 0 \text{ in the corners of } \bar{E}\}. \end{aligned}$$

By our assumption that $s > 2$ and by Sobolev's embedding theorem, $X_\|$ is well-defined. Obviously, $X_\|$ is a closed subspace of $H^s(E, \mathbb{R}^2)$. We now show that for each $\eta \in \mathrm{Emb}^*(\bar{E})$

$$\Phi_\eta^{-1}(U_\eta \cap X_\|) = \mathrm{Emb}^*(\bar{E}) \cap (\eta + U_\eta) \tag{27}$$

holds true. This will prove that $\mathrm{Emb}^*(\bar{E})$ is a submanifold of $\mathrm{Emb}(\bar{E})$:

If $\zeta \in \mathrm{Emb}^*(\bar{E}) \cap (\eta + U_\eta)$, then Σ_0 and Σ_1 are mapped onto themselves by ζ and the four corner points stay fixed. The same holds true for η, therefore $\eta - \zeta$ is tangential to Σ_0 and Σ_1 and vanishes in the corner points, i.e. $\eta - \zeta \in X_\|$.

If, on the other hand, $X \in U_\eta \cap X_\|$, then $\eta + X \in \mathrm{Emb}(\bar{E})$ fixes the four corners, because η does so and X vanishes in the corners. Because X is tangential to Σ_0 and Σ_1, $\eta + X$ maps Σ_0 and Σ_1 onto segments of straight lines. Since $\eta + X$ is an embedding which fixes the corners, it then follows, that Σ_0 and Σ_1 are mapped bijectively onto themselves. Therefore, $\eta + X \in \mathrm{Emb}^*(\bar{E})$. This shows that Eq. (27) holds true and proves our claim that $\mathrm{Emb}^*(\bar{E})$ is a submanifold of $\mathrm{Emb}(\bar{E})$. □

Furthermore,

$$T_\eta \,\mathrm{Emb}^*(\bar{E}) = X_\| \tag{28}$$

holds. This is easy to see: If $X \in T_\eta \,\mathrm{Emb}^*(\bar{E})$, then by definition there is a one-parameter family $\eta_t \in \mathrm{Emb}^*(\bar{E})$, $\eta_0 = \eta$, such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \eta_t = X$$

and $X \in X_{\|}$ obviously holds. If, one the other hand, $X \in X_{\|}$, then $\eta_t := \eta + tX$ is in $\mathrm{Emb}(\bar{E})$ for $t \in \mathbb{R}$ sufficiently small, because this set is open in $H^s(E, \mathbb{R}^2)$ as we showed above. Since η_t fixes the plates and the corners of E, one has $\eta_t \in \mathrm{Emb}^*(\bar{E})$ and furthermore $(\mathrm{d}/\mathrm{d}t)\eta_t = X$ at $t = 0$. This proves (28). We will now show that a differentiable structure can be defined on the configuration space $\mathcal{C}$ of the liquid drop between the two plates.

Theorem 3. *$\mathcal{C}$ is a submanifold of $\mathrm{Emb}^*(\bar{E})$.*

Proof. Before giving the details of the proof we will outline its strategy. For $l \in \mathbb{N}$ let $H^l(\Lambda^2)$ denote the vector space of volume elements on E of class H^l. The space $H^l(\Lambda^1)$ is defined analogously. The volume elements are of the form $f\ \mathrm{d}x \wedge y$ for some function $f \in H^l(E, \mathbb{R})$. We will prove that

$$\Psi : \mathrm{Emb}^*(\bar{E}) \to W, \qquad \eta \mapsto \eta^* \,\mathrm{d}x \wedge \mathrm{d}y,$$

is smooth, where W denotes a certain subspace of $H^{s-1}(\Lambda^2)$ to be defined later. Then we will show that

$$\mathcal{C} = \Psi^{-1}(\mathrm{d}x \wedge \mathrm{d}y)$$

is a submanifold of $\mathrm{Emb}^*(\bar{E})$ by proving that the map $T_\eta\Psi$ is surjective at every $\eta \in \Psi^{-1}(\mathrm{d}x \wedge \mathrm{d}y)$. Note that

$$\Psi(\eta) = (\det D\eta)\,\mathrm{d}x \wedge \mathrm{d}y.$$

Now we will fill in the details in this outline of the proof. For $l > 1 H^l(E, \mathbb{R})$ is a ring under pointwise multiplication, the so-called Schauder ring. This result is stated in [2] for the case that E has a smooth boundary. Using the extention theorem of Calderon and Zygmund one sees that it also holds in the case of a square-shaped E. Therefore, $\det D\eta \in H^{s-1}(E, \mathbb{R})$ for $\eta \in H^s(E, \mathbb{R}^2)$. Obviously, Ψ is a smooth map.

Note that

$$\eta^*(\mathrm{d}x \wedge \mathrm{d}y) = \eta^*\,\mathrm{d}(x\,\mathrm{d}y) = \mathrm{d}(\eta^*(x\,\mathrm{d}y)) \tag{29}$$

is an exact form. For $\eta \in \mathrm{Emb}^*(\bar{E})$, the two-form $\eta^*(\mathrm{d}x \wedge \mathrm{d}y)$ has some additional properties as we will show now. For $\eta = (\eta_1, \eta_2)^{\mathrm{T}}$, one has

$$\eta^*(x\,\mathrm{d}y) = \eta_1\,\mathrm{d}\eta_2 = \eta_1 \partial_x \eta_2\,\mathrm{d}x + \eta_1 \partial_y \eta_2\,\mathrm{d}y. \tag{30}$$

Because η keeps the contact surfaces Σ_0 and Σ_1 between drop and plates fixed, one has $\eta_2 = 0$ on $\Sigma_0 \cup \Sigma_1$. Making use of this in (30) one gets

$$\eta^*(x\,\mathrm{d}y) = \eta_1 \partial_y \eta_2\,\mathrm{d}y \quad \text{on } \Sigma_0 \cup \Sigma_1. \tag{31}$$

Let

$$i_{\Sigma_0} : \Sigma_0 \to \bar{E} \quad \text{and} \quad i_{\Sigma_1} : \Sigma_1 \to \bar{E}$$

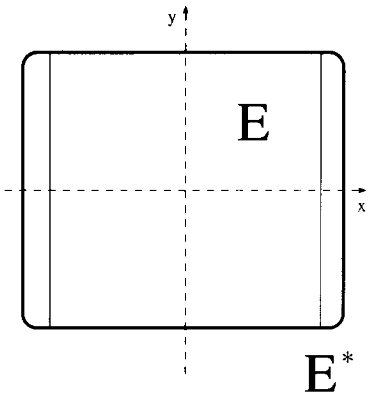

Fig. 4. Smoothing the boundary of E.

denote the canonical embeddings. From (31) it follows that

$$i_{\Sigma_i}(\eta^*(x\,\mathrm{d}y)) = 0 \quad \text{for } i = 0, 1. \tag{32}$$

By definition, this means that the one-form $\eta^*(x\,\mathrm{d}y)$ is *normal* to Σ_0 and to Σ_1. From (29) we see that

$$\eta^*(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}\beta, \tag{33}$$

for a one-form β of class H^{s-1}. This one-form is not uniquely determined and we now show that we can find $\beta \in H^s(\Lambda^1)$ satisfying Eq. (33) and also

$$i_{\Sigma_i}(\beta) = 0 \quad \text{for } i = 0, 1. \tag{34}$$

From Eq. (30) one has

$$\eta^*(\mathrm{d}x \wedge \mathrm{d}y) = (\partial_x\eta_1\partial_y\eta_2 - \partial_y\eta_1\partial_x\eta_2)\,\mathrm{d}x \wedge \mathrm{d}y.$$

By the Calderon–Zygmund theorem there is a function $f_\eta \in H^{s-1}(\mathbb{R}^2, \mathbb{R})$, such that

$$f_\eta = (\partial_x\eta_1\partial_y\eta_2 - \partial_y\eta_1\partial_x\eta_2) \quad \text{in } E.$$

Extend E to a domain E^* with smooth boundary as shown in Fig. 4. By multiplying f_η with a smooth bump function that has support in $E^*\backslash E$ we can construct a function $\bar{f} \in H^{s-1}(\mathbb{R}^2, \mathbb{R})$ with $\bar{f} = f_\eta$ in E and $\int_{E^*} \bar{f}\,\mathrm{d}A = 0$. Let $\bar{g} \in H^{s+1}(E, \mathbb{R})$ be any solution of

$$\Delta\bar{g} = \bar{f}, \qquad \frac{\partial\bar{g}}{\partial n} = 0 \quad \text{on } \partial\bar{E}^*. \tag{35}$$

This Neumann problem is solvable by construction of $\bar{f}$. Let $g \in H^{s+1}(E, \mathbb{R})$ denote the restriction of $\bar{g}$ to E and define $\bar{\beta} \in H^s(\Lambda^1)$ by

$$\bar{\beta} = -\partial_y g\,\mathrm{d}x + \partial_x g\,\mathrm{d}y. \tag{36}$$

(We note, that in the three-dimensional case one has to define $\bar{\beta}$ by $\bar{\beta} = g_z\,\mathrm{d}x \wedge \mathrm{d}y + g_x\,\mathrm{d}y \wedge \mathrm{d}z - g_y\,\mathrm{d}x \wedge \mathrm{d}z$. Then $\mathrm{d}\bar{\beta} = \Delta g\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ holds, analogously to the two-dimensional case.) One has

$$\begin{aligned} \mathrm{d}\bar{\beta} &= \partial_{yy} g\,\mathrm{d}x \wedge \mathrm{d}y + \partial_{xx} g\,\mathrm{d}x \wedge \mathrm{d}y \\ &= \eta^*(\mathrm{d}x \wedge \mathrm{d}y), \end{aligned} \tag{37}$$

and $i_{\Sigma_i}(\bar{\beta}) = 0$ for $i = 0, 1$, i.e. Eqs. (33) and (34) are satisfied by $\bar{\beta} \in H^s(\Lambda^1)$. Note that these properties are conserved if we add to $\bar{\beta}$ the differential of a function $h \in H^{s+1}(\mathbb{R}^2, \mathbb{R})$ with

$$\partial_x h = 0 \quad \text{on } \Sigma_0 \cup \Sigma_1. \tag{38}$$

We will now show, that such a function $h \in H^{s+1}(\mathbb{R}^2, \mathbb{R})$ exists, having the additional property

$$\mathrm{d}h + \bar{\beta} = 0, \tag{39}$$

i.e.

$$\begin{pmatrix} \partial_x h \\ \partial_y h \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{\beta}_2 \end{pmatrix} = 0, \tag{40}$$

at the four corner points for E. Using the Calderon–Zygmund theorem, we extend $\bar{\beta} = (\bar{\beta}_1, \bar{\beta}_2)^{\mathrm{T}}$ to a function in $H^s(\mathbb{R}^2, \mathbb{R}^2)$ which we also call $\bar{\beta}$. Let

$$h^1(x, y) = (1 - y)\tfrac{1}{2}[(\bar{\beta}_2(1, 1) - \bar{\beta}_2(-1, 1))x + (\bar{\beta}_2(1, 1) + \bar{\beta}_2(-1, 1))].$$

Then

$$h^1 + \partial_x h^1 = 0 \quad \text{on } \Sigma_0,$$

and

$$\partial_y h^1 + \bar{\beta}_2 = 0 \quad \text{for } (x, y) = (-1, 1), (1, 1).$$

Define h^2 analogously such that

$$h^2 = \partial_x h^2 = 0 \quad \text{on } \Sigma_1,$$

and

$$\partial_y h^2 + \bar{\beta}_2 = 0 \quad \text{for } (x, y) = (-1, -1), (1, -1).$$

Now introduce a bump function $B \in C^\infty(\mathbb{R}, \mathbb{R})$, satisfying

$$B(s) = 1 \quad \text{for } s \in (-\tfrac{1}{4}, \tfrac{1}{4}) \quad \text{and} \quad B(s) = 0 \quad \text{for } s \notin (-\tfrac{1}{2}, \tfrac{1}{2}).$$

Define

$$h(x, y) = B(y - 1)h^1(x, y) + B(y + 1)h^2(x, y).$$

Then, Eq. (38) is satisfied, because h vanishes on $\Sigma_0 \cup \Sigma_1$. Also, Eq. (40) holds by construction. Therefore,

$$\beta = \bar{\beta} + \mathrm{d}h$$

satisfies

$$\mathrm{d}\beta = \eta^*(\mathrm{d}x \wedge \mathrm{d}y), \tag{41}$$

$$\beta \in H^s(\Lambda^1), \tag{42}$$

$$i_{\Sigma_0}(\beta) = 0, \tag{43}$$

$$i_{\Sigma_1}(\beta) = 0, \tag{44}$$

$$\beta = 0 \quad \text{at the edges of } E. \tag{45}$$

Let

$$W = \{\,\mathrm{d}\beta, \beta \text{ satisfies (42)–(45)}\}. \tag{46}$$

(Note that by replacing the function f_η by an arbitrary function $f \in H^{s-1}(\mathbb{R}^2, \mathbb{R})$ the argument above actually proves that $W = H^{s-1}(\Lambda^2)$.) We have shown that

$$\Psi(\mathrm{Emb}^*(\bar{E})) \subseteq W \tag{47}$$

and that we can consider Ψ as a map

$$\Psi : \mathrm{Emb}^*(\bar{E}) \to W, \qquad \eta \mapsto \eta^*\,\mathrm{d}x \wedge \mathrm{d}y. \tag{48}$$

It is easy to see that Ψ is differentiable. Now we want to show that

$$W \subseteq T_\eta\Psi(T_\eta\,\mathrm{Emb}^*(\bar{E})). \tag{49}$$

This will prove that $T_\eta\Psi$ is surjective and that therefore

$$C = \Psi^{-1}(\mathrm{d}x \wedge \mathrm{d}y)$$

is a submanifold of $\mathrm{Emb}^*(\bar{E})$.

By [2] one has

$$T_\eta\Psi X = \eta^*(L_{X\circ\eta^{-1}}(\mathrm{d}x \wedge \mathrm{d}y)), \tag{50}$$

for a vector field X. We verify Eq. (50) by a direct computation.

Because $\mathrm{d}x \wedge \mathrm{d}y$ is a closed form,

$$L_{X\circ\eta^{-1}}\,\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}i_{X\circ\eta^{-1}}(\mathrm{d}x \wedge \mathrm{d}y). \tag{51}$$

holds. By definition of the contraction operator one has for two vector fields v and w

$$i_w(\mathrm{d}x \wedge \mathrm{d}y)v = \mathrm{d}x \wedge \mathrm{d}y(w, v) = w_1 v_2 - w_2 v_1,$$

and therefore

$$i_w(\mathrm{d}x \wedge \mathrm{d}y) = -w_2\,\mathrm{d}x + w_1\,\mathrm{d}y,$$

and

$$\begin{aligned} \mathrm{d}i_{X\circ\eta^{-1}}(\mathrm{d}x \wedge \mathrm{d}y) &= \mathrm{d}(-X_2\circ\eta^{-1}\,\mathrm{d}x + X_1\circ\eta^{-1}\,\mathrm{d}y) \\ &= (\partial_y(X_2\circ\eta^{-1}) + \partial_x(X_1\circ\eta^{-1}))\,\mathrm{d}x \wedge \mathrm{d}y. \end{aligned} \tag{52}$$

Because

$$\begin{pmatrix} \partial_x(\eta^{-1})_1 & \partial_y(\eta^{-1})_1 \\ \partial_x(\eta^{-1})_2 & \partial_y(\eta^{-1})_2 \end{pmatrix} = \frac{1}{\det D\eta}\begin{pmatrix} \partial_y\eta_2 & -\partial_y\eta_1 \\ -\partial_x\eta_2 & \partial_x\eta_1 \end{pmatrix},$$

one has

$$\begin{aligned} \partial_y(X\circ\eta^{-1})_2 &= \partial_y(X_2\circ\eta^{-1}) \\ &= \partial_x X_2\partial_y(\eta^{-1})_1 + \partial_y X_2\partial_y(\eta^{-1})_2 \\ &= \frac{1}{\det D\eta}(\partial_x X_2(-\partial_y\eta_1) + \partial_y X_2\partial_x\eta_1), \end{aligned}$$

and

$$\begin{aligned} \partial_x(X\circ\eta^{-1})_1 &= \partial_x(X_1\circ\eta^{-1}) \\ &= \partial_x X_1\partial_x(\eta^{-1})_1 + \partial_y X_1\partial_x(\eta^{-1})_2 \\ &= \frac{1}{\det D\eta}(\partial_x X_1\partial_y\eta_2 + \partial_y X_1(-\partial_x\eta_2)). \end{aligned}$$

Thefore, using Eqs. (51) and (52),

$$\begin{aligned} &L_{X\circ\eta^{-1}}\,\mathrm{d}x \wedge \mathrm{d}y \\ &\quad = \frac{1}{\det D\eta}(\partial_x X_1\partial_y\eta_2 + \partial_y X_2\partial_x\eta_1 - \partial_y X_1\partial_x\eta_2 - \partial_x X_2\partial_y\eta_1)\,\mathrm{d}x \wedge \mathrm{d}y, \end{aligned}$$

and

$$\begin{aligned} \eta^* L_{X\circ\eta^{-1}}\,\mathrm{d}x \wedge \mathrm{d}y = (&\partial_x X_1\partial_y\eta_2 + \partial_y X_2\partial_x\eta_1 \\ &-\partial_y X_1\partial_x\eta_2 - \partial_x X_2\partial_y\eta_1)\,\mathrm{d}x \wedge \mathrm{d}y. \end{aligned} \tag{53}$$

But because, by definition of Ψ,

$$\Psi(\eta) = \eta^*(\mathrm{d}x \wedge \mathrm{d}y) = (\partial_x\eta_1\partial_y\eta_2 - \partial_y\eta_1\partial_x\eta_2)\,\mathrm{d}x \wedge \mathrm{d}y,$$

Eq. (50) follows from Eq. (53). If, in particular, η =id, the identify map, then by (50) and (51),

$$T_{\mathrm{id}}\Psi X = \mathrm{d}i_X(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}(-X_2\,\mathrm{d}x + X_1\,\mathrm{d}y). \tag{54}$$

To see that (49) holds at η =id, take any $w \in W$ choose $\beta = \beta_1\,\mathrm{d}x + \beta_2\,\mathrm{d}y$, such that $w = \mathrm{d}\beta$ and (42)–(45) are satisfied and define $X \in H^s(E, \mathbb{R}^2)$, $X = (X_1, X_2)$, by

$$\begin{pmatrix} X_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \beta_2 \\ -\beta_1 \end{pmatrix}. \tag{55}$$

Then, $\langle X, (0, 1)^{\mathrm{T}}\rangle = 0$ on Σ_i, $i = 0, 1$, and because β vanishes in the four corners of E, the same holds true for X. From (554) it follows that

$$T_{\mathrm{id}}\Psi X = w.$$

To see that Eq. (49) holds for an arbitrary η, take any $w \in W$ and analogously to what we did above determine $\beta = \beta_1\,\mathrm{d}x + \beta_2\,\mathrm{d}y$, such that $(\eta^{-1})^* w = \mathrm{d}\beta$, $\beta \in H^s(\Lambda^1(\eta(E)))$, and (43)–(45) are satisfied. As in the case η =id, we define a vector field $\bar{X} \in H^s(\eta(E), \mathbb{R}^2)$ such that

$$L_{\bar{X}}(\mathrm{d}x \wedge \mathrm{d}y) = (\eta^{-1})^* w,$$

and therefore

$$\eta^* L_{\bar{X}}(\mathrm{d}x \wedge \mathrm{d}y) = w.$$

Then, putting $X = \bar{X} \circ \eta$, one has

$$\eta^* L_{X \circ \eta^{-1}}(\mathrm{d}x \wedge \mathrm{d}y) = w.$$

Because $\langle X, (0, 1)^{\mathrm{T}}\rangle = 0$ on Σ_i, $i = 0, 1$, β as well as X vanish in the four corners of E and $X \in H^s(E, \mathbb{R}^2)$ one has $X \in T_\eta \mathrm{Emb}^*(\bar{E})$ and

$$T_\eta \Psi X = w.$$

This finishes our proof that the configuration space $\mathcal{C}$ of the drop is a differentiable manifold. □

To define a manifold structure on the configuration space of a liquid bridge with moving contact lines between two infinitely extended plates, the following modifications have to be made in the proof above (again considering the two-dimensional case only):

One drops the assumption of fixed contact points and considers the subset $\mathrm{Emb}^+(\bar{E})$ of $\mathrm{Emb}(\bar{E})$, consisting of embeddings that map the contact lines (in the three-dimensional case: the contact surfaces) of the reference configuration with plate P_i into P_i, $i = 0, 1$. Just as above one proves that $\mathrm{Emb}^+(\bar{E})$ is a submanifold of $\mathrm{Emb}(\bar{E})$ and that the set of volume-preserving maps in $\mathrm{Emb}^+(\bar{E})$ is a submanifold $\mathrm{Emb}^+_{\mathrm{vol}}(\bar{E})$ of $\mathrm{Emb}^+(\bar{E})$. The configuration space $\mathcal{C}_m$ of the drop with moving contact lines is then defined as the set of those maps $\eta \in \mathrm{Emb}^+_{\mathrm{vol}}(\bar{E})$ that have the following properties: The free boundary F_E of the reference configuration is mapped by η into the region between the two plates and $\eta(F_E)$ touches plates P_i only in the images of the contact points of F_E with plate P_i, $i = 0, 1$. Furthermore, the contact angles in which $\eta(F_E)$ meets the two plates are assumed to be different from zero. Then by (25) the set $\mathcal{C}_m$ is an open subset of $\mathrm{Emb}^+_{\mathrm{vol}}(\bar{E})$ and therefore a submanifold of $\mathrm{Emb}^+_{\mathrm{vol}}(\bar{E})$. As mentioned in Section 1, motions in which the free boundary of the drop hits a plate cannot be described with this model of configuration space for a drop with moving contact lines between infinitely extended plates. We note that the values of the contact angles in which the free boundary of the drop meets the two plates are not specified in the definition of $\mathcal{C}_m$. In [6] the exact values of the contact angles are determined by the choice of the Hamiltonian which governs the dynamics of the liquid bridge.

4. Summary of some results on the stability and bifurcation of rigidly rotating cylindrical liquid drops

In this section we will summarize results of Kruse [7] on the stability and bifuration of rigidly rotating liquid cylinders with fixed contact lines. Liquid cylinders with height h and base radius d rotating with angular velocity ω represent solutions to the equations of motion (9)–(14) for any value of $\omega \in \mathbb{R}$. In the companion paper [7] we use a variant of the energy–momentum method of Simo, Lewis, Marsden and their co-workers (for references to the literature see Section 1) to study the stability and bifurcation behaviour of these solutions with respect to axisymmetric perturbations of the drop shape. As a bifurcation parameter we use on the one hand the angular velocity ω of the rotating drop and on the other hand its angular momentum μ. The discussion is analogous to the one given in [9], were liquid bridges with free contact lines, but fixed contact angles were considered. To be more specific, let

$$f : [0, h] \to \mathbb{R}_0^+$$

parameterize the profile curve of an axisymmetric drop with free boundary Σ_f. Let $\mathrm{Vol}(f)$ denote the drop volume, $V(f) = \tau \int_{\Sigma_f} \mathrm{d}A$ its potential energy and $I(f)$ its moment of inertia about the z-axis. A rigidly rotating axisymmetric drop can be characterized as a critical point of the functional

$$V(f) - \tfrac{1}{2}\omega^2 I(f) - c\mathrm{Vol}(f),$$

for some $c \in \mathbb{R}$. The functional

$$V_\omega = V(f) - \tfrac{1}{2} I(f)$$

is called the *augmented potential* of the drop (see [12]). The drop with profile $f : [0, h] \to \mathbb{R}^+$ and angular velocity $\omega \in \mathbb{R}$ is orbitally stable with respect to axisymmetric perturbations if the second derivative $D^2(V_\omega - c\mathrm{Vol})(f)(\delta f, \delta g)$ is definite with respect to variations $\delta f, \delta g : [0, h] \to \mathbb{R}$ that satisfy the linearized volume constraint

$$\int_0^h \delta f \,\mathrm{d}z = \int_0^h \delta g \,\mathrm{d}z = 0.$$

In [7] we show that a cylindrical drop with base radius d and angular velocity ω about the z-axis is stable if

$$\frac{4\pi^2 d\tau}{h^2} > \frac{\tau}{d} + \omega^2 d^2.$$

This inequality is violated for $\omega^2 > \omega_0^2$, where ω_0 is a solution of

$$\frac{h^2}{4\pi^2 d^2} + \frac{\omega_0^2 h^2 d}{4\tau\pi^2} = 1. \tag{56}$$

At the critical angular velocity ω_0 a subcritical pitchfork bifurcation occurs in the sense that non-cylinderical solutions exist for $\omega^2 < \omega_0^2$. There is a second way to characterize rigidly rotating axisymmetric drops. Let

$$V_\mu(f) = V(f) + \frac{1}{2}\frac{\mu^2}{I(f)}$$

denote the *amended potential* of the drop (compare [12]). Axisymmetric drops with profile $f : [0, h] \to \mathbb{R}^+$ rotating rigidly about the z-axis with angular velocity ω can be characterized as critical points of the functional

$$V_\mu - c\mathrm{Vol},$$

for some $c \in \mathbb{R}$. These rigidly rotating drops are orbitally stable with respect to axisymmetric perturbations of the drop shape if the second derivative $D^2(V_\mu - c\mathrm{Vol})(f)(\delta f, \delta g)$ is definite for variations $\delta f, \delta g : [0, h] \to \mathbb{R}^+$ which satisfy the linearized volume constraint. In [7] it is shown that using the amended potential in the stability analysis of rigidly rotating liquid cylinders one arrives at the same stability result as in the case of the augmented potential, i.e. the liquid cylinder with base radius d is stable of $|\mu| < |\mu_0|$, where

$$\mu_0 = I(\mathrm{d})\omega_0.$$

If one uses the angular momentum μ as a bifurcation parameter instead of the angular velocity and characterizes rigidly rotating drops as critical points of the amended potential instead of the augmented potential then a pitchfork bifurcation takes place at the parameter value $\mu = \mu_0$. Let $\sigma = h^2/4\,\mathrm{d}^2$ and

$$\sigma_1 = \frac{\pi^2}{324}(147 - \sqrt{15777}), \qquad \sigma_2 = \frac{\pi^2}{324}(147 + \sqrt{15777}).$$

For $\sigma \in (0, \sigma_1) \cup (\sigma_2, \infty)$ the bifurcation is subcritical. However, if $\sigma \in (\sigma_1, \sigma_2)$, the bifurcation is supercritical and solutions on the bifurcating branches are stable (compare [1]). Note that σ is up to a constant factor just the square of the aspect ratio of the drop.

Acknowledgements

This research is supported by the DFG under the contract Sch 233/3-2. The author would like to thank Jürgen Scheurle for fruitful discussions on the subject of this paper.

References

[1] H. Amann, Gewöhnliche Differentialgleichungen, de Gruyter, Berlin, 1983.
[2] D.G. Ebin, J.E. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, Ann. Math. 92 (1970) 102–163.
[3] W. Eidel, Weak non-linear axisymmetric oscillations of an inviscid liquid column with anchored free surface under zero-gravity, Micrograviry Sci. Technol. VII/1 (1994) 6–11.
[4] R. Finn, Equilibrium Capillary Surfaces, Springer, New York, 1986.
[5] J. Jost, Riemannian Geometry and Geometric Analysis, Springer, Berlin, 1995.
[6] H.-P. Kruse, Flüssigkeitstropfen zwischen parallelen Platten: Hamiltonsche Struktur, Existenz von Lösungen und Stabilität, Doctoral Thesis, Universität Hamburg, 1992.
[7] H.-P. Kruse, Bifurcation of rotating inviscid liquid bridges with fixed contact lines, Preprint, 1998.
[8] H.-P. Kruse, J.E. Marsden, J. Scheurle, On uniformly rotating fluid drops trapped between two parallel plates, in: (Eds.), E.L. Allgower, K. Georg, R. Miranda Exploiting Symmetry in Applied and Numerical Analysis, 1992 AMS-SIAM Summer Seminar in Applied Mathematics, 26 July–1 August 1992, Colorado State University, Providence, Rhode Island, American Mathematical Society.
[9] H.-P. Kruse, J. Scheurle, On the bifurcation and stability of rigidly rotating inviscid liquid bridges, J. Nonlinear Sci. 8 (1998) 215–232.
[10] D. Lewis, J.E. Marsden, R. Montgomery, T. Ratiu, The Hamiltonian structure for dynamic free boundary value problems, Physica D 18 (1986) 391–404.
[11] J.E. Marsden, J.C. Simo, D. Lewis, T.A. Posbergh, A block diagonalization theorem in the energy momentum method, Cont. Math. AMS 97 (1989) 297–313.
[12] J.E.Marsden, Lectures on Mechanics, London Math. Soc. Lect. Note Ser., vol. 174, Cambridge University Press, Cambridge, 1992.
[13] J.C. Simo, D. Lewis, J.E. Marsden, Stability of relative equilibria. Part I: The reduced energy momentum method, Arch. Rat. Mech. Anal. 115 (1991) 15–69.
[14] J. Wloka, Partielle Differentialgleichungen, Teubner, Stuttgart, 1982.